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DEPARTMENT OF NUMERICAL MATHEMATICS NW 29/76 MARCH

H.J.J. TE RIELE

ON INTEGER ARITHMETIC PROGRESSIONS OF LENGTH FOUR

Prepublication

amsterdam 1976

stichting
mathematisch
centrum

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2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 10A40

On integer arithmetic progressions of length four *)

by

H.J.J. te Riele

ABSTRACT

Let $t_i(n)$ be the number of pairs (a,b) ($a,b \in \mathbb{N}$, $1 \le a,b \le n$, $a \ne b$) which belong to i integer arithmetic progressions of length four with positive terms $\le n$.

In this note it is shown that $t_i(n) \sim \frac{C_i}{1260} n^2$ $(n \to \infty)$, where $C_0 = 280$, $C_1 = 324$, $C_2 = 214$, $C_3 = 189$, $C_4 = 106$, $C_5 = 105$, $C_6 = 42$ and $C_{>6} = 0$.

KEY WORDS & PHRASES: *Arithmetic progressions*

*) This paper is not for review; it is meant for publication elsewhere

1. RESULTS

Let n, a and b be positive integers such that $1 \le a,b \le n$, $a \ne b$. Define $f_n(a,b)$ as the number of integer arithmetic progressions of length *four* with positive terms $\le n$ to which a and b belong (in this order). The possible values of $f_n(a,b)$ are $0,1,\dots,6$. We denote by $t_i(n)$ the number of pairs (a,b) for which $f_n(a,b)$ takes the value i.

In this note it is proved that for $n \to \infty$(*) $t_i(n) \sim \frac{C_i}{1260} n^2$, where $C_0 = 280$, $C_1 = 324$, $C_2 = 214$, $C_3 = 189$, $C_4 = 106$, $C_5 = 105$ and $C_6 = 42$. Moreover, it is proved that the limit of the average value of $f_n(a,b)$ is 2(as $n \to \infty$).

Analogous results for arithmetic progressions of length *three* were obtained by DRESSLER [1]. In principle, our method is a formalization of DRESSLER's method. It can be extended to arithmetic progressions of length greater than four, but the amount of work will be a very rapidly increasing "function" of the length.

2. PREPARATORY CALCULATIONS

Let Ω_n be the set of pairs (a,b) of integers a,b such that $1 \le a < b \le n$. Choose $(a,b) \in \Omega_n$. A necessary and sufficient condition for a and b to be the *first* and the *second* term, respectively, of an integer arithmetic progression of length four between 1 and n, inclusive, is that $b + 2(b-a) \le n$. A necessary and sufficient condition for a and b to be the *first* and the *third* term, respectively, is that $(2|b-a) \wedge (b+(b-a)/2 \le n)$, and so on. The *six* essentially different possibilities and the corresponding conditions are given in Table 1. The conditions are denoted by $c_1, c_2, \dots, c_6$, in the order indicated in the table.

(*) If g(n) and h(n) are defined and positive for all $n \in \mathbb{N}$, then by $g(n) \sim h(n)$ we mean $\lim_{n\to\infty} g(n)/h(n) = 1$, and we read: g(n) is asymptotic to h(n).

TABLE 1

The six necessary and sufficient conditions $c_1,\ldots,c_6$

first term	second term	third term	fourth term	necessary and sufficient condition
a	b	$2b - a$	$3b - 2a$	$3b - 2a \le n$ (c_1)
a	$a + \frac{b-a}{2}$	b	$b + \frac{b-a}{2}$	$(2\mid b-a) \wedge (b+ \frac{b-a}{2} \le n)$ (c_3)
a	$a + \frac{b-a}{3}$	$a + 2\frac{b-a}{3}$	b	$3\mid b-a$ (c_6)
$2a - b$	a	b	$2b - a$	$(2a-b\ge 1) \wedge (2b-a\le n)$ (c_5)
$a - \frac{b-a}{2}$	a	$a + \frac{b-a}{2}$	b	$(2\mid b-a) \wedge (a- \frac{b-a}{2} \ge 1)$ (c_4)
$3a - 2b$	$2a - b$	a	b	$3a - 2b \ge 1$ (c_2)

Let $V_1, V_2, \ldots, V_8$ be subsets of Ω_n, the elements of which satisfy, respectively, the following conditions

$$(2.1) \quad \begin{cases} 3b - a \le 2n \; (V_1), \quad 2b - a \le n \; (V_2), \quad 3b - 2a \le n \; (V_3), \\ 3a - b \ge 2 \; (V_4), \quad 2a - b \ge 1 \; (V_5), \quad 3a - 2b \ge 1 \; (V_6), \\ 2\mid b - a \; (V_7), \quad 3\mid b - a \; (V_8). \end{cases}$$

Then the pairs $(a,b) \in \Omega_n$ satisfying c_1 belong to V_3, the pairs satisfying c_2 belong to V_6, and so on:

(2.2)

condition on (a,b)	c_1	c_2	c_3	c_4	c_5	c_6
(a,b) belongs to	V_3	V_6	$V_1 \cap V_7$	$V_4 \cap V_7$	$V_2 \cap V_5$	V_8

In the sequel, the *negation* of c_i will be denoted by $\bar{c}_i$: for instance, $\bar{c}_3$ is the condition $(2 \nmid b-a) \vee (b+\frac{b-a}{2} > n)$. The *complement* of a set V_i (with respect to Ω_n) is denoted by $\bar{V}_i$ or $V_{\bar{i}}$. The intersection $\cap_{j=1}^{k} V_{i_j}$ of k sets $V_{i_1}, V_{i_2}, \ldots, V_{i_k}$ will be denoted by $V_{i_1,i_2,\ldots,i_k}$: for instance, $V_{1,\bar{3},7} = V_1 \cap \bar{V}_3 \cap V_7$.

In order to find an asymptotic estimate for $t_i(n)$, we shall determine all *disjoint* sets of pairs (a,b) which can be formed by the conjunction of i conditions out of $c_1, c_2, \ldots, c_6$ *set true* with the remaining 6-i conditions *set false*. This yields $2^6 = 64$ different sets. Since c_6 (resp. $\bar{c}_6$) contributes a factor 1/3 (resp. 2/3) to the estimates, we need only determine the 32 sets which remain after dropping the conditions c_6 and $\bar{c}_6$. These sets will be denoted by $W_0, W_1, \ldots, W_{31}$. The index k of W_k corresponds in the following way to the conditions to be satisfied by the elements of W_k: the j-th binary digit of k, counted from the left, is 0 or 1 according to whether the elements of W_k *do* or *do not* satisfy c_j (j=1,2,3,4,5). For example: $22_{10} = 10110_2$, so that

$$W_{22} = \{(a,b) \in \Omega_n \mid \bar{c}_1 \wedge c_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \wedge c_5\}.$$

From (2.1) it follows that

(2.3) $$V_3 \subset V_2 \subset V_1 \text{ and } V_6 \subset V_5 \subset V_4,$$

so that, using this and (2.2), we obtain

$$\begin{aligned} W_{22} &= \bar{V}_3 \cap V_6 \cap (\overline{V_1 \cap V_7}) \cap (\overline{V_4 \cap V_7}) \cap V_2 \cap V_5 \\ &= V_2 \cap \bar{V}_3 \cap (\bar{V}_1 \cup \bar{V}_7) \cap (\bar{V}_4 \cup \bar{V}_7) \cap V_6 \\ &= \left(V_{\bar{1},2,\bar{3}} \cup V_{2,\bar{3},\bar{7}}\right) \cap \left(V_{\bar{4},6} \cup V_{6,\bar{7}}\right) \\ &= V_{2,\bar{3},\bar{7}} \cap V_{6,\bar{7}} = V_{2,\bar{3},6,\bar{7}}. \end{aligned}$$

All sets W_k ($k=0,1,\ldots,31$) were determined in this way, and tabulated in Table 2 ($\emptyset$ denotes the empty set).

TABLE 2

The sets $W_0, W_1, \ldots, W_{31}$

k(decimal, binary)		W_k	k(decimal, binary)		W_k
0,	00000	$V_{3,6,7}$	16,	10000	$V_{2,\bar{3},6,7}$
1,	00001	$\emptyset$	17,	10001	$V_{1,\bar{2},6,7}$
2,	00010	$\emptyset$	18,	10010	$\emptyset$
3,	00011	$\emptyset$	19,	10011	$\emptyset$
4,	00100	$\emptyset$	20,	10100	$\emptyset$
5,	00101	$\emptyset$	21,	10101	$V_{\bar{1},6,7}$
6,	00110	$V_{3,6,\bar{7}}$	22,	10110	$V_{2,\bar{3},6,\bar{7}}$
7,	00111	$\emptyset$	23,	10111	$V_{\bar{2},6,\bar{7}}$
8,	01000	$V_{3,5,\bar{6},7}$	24,	11000	$V_{2,\bar{3},5,\bar{6},7}$
9,	01001	$V_{3,4,\bar{5},7}$	25,	11001	$V_{1,\bar{2},4,\bar{6},\bar{7}} \cup V_{1,\bar{3},4,\bar{5},\bar{7}}$
10,	01010	$\emptyset$	26,	11010	$\emptyset$
11,	01011	$V_{3,\bar{4},7}$	27,	11011	$V_{1,\bar{3},\bar{4},7}$
12,	01100	$\emptyset$	28,	11100	$\emptyset$
13,	01101	$\emptyset$	29,	11101	$V_{\bar{1},4,\bar{6},7}$
14,	01110	$V_{3,5,\bar{6},\bar{7}}$	30,	11110	$V_{2,\bar{3},5,\bar{6},\bar{7}}$
15,	01111	$V_{3,\bar{5},\bar{7}}$	31,	11111	$V_{\bar{1},\bar{4}} \cup V_{\bar{2},\bar{6},\bar{7}} \cup V_{\bar{3},\bar{5},\bar{7}}$

Now we shall determine asymptotic estimates for the *number of elements* in the sets $W_0, W_1, \ldots, W_{31}$. Let the number of elements in a set S be denoted by $|S|$. We first notice that both V_7 and $\bar{V}_7$ contribute a factor $\frac{1}{2}$ to the estimates. For instance, $|W_0| = |V_{3,6,7}| \sim \frac{1}{2}|V_{3,6}|$. Furthermore, from (2.1) one may derive the following permutation property: The number of elements in a set S_1, which is the intersection of some sets from the collection $\{V_1, V_2, V_3, V_4, V_5, V_6, \bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4, \bar{V}_5, \bar{V}_6\}$, *equals* the number of elements in the set S_2 which is obtained from S_1 after replacing

$$V_1, V_2, V_3, V_4, V_5, V_6, \bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4, \bar{V}_5, \bar{V}_6$$

by

$$V_4, V_5, V_6, V_1, V_2, V_3, \bar{V}_4, \bar{V}_5, \bar{V}_6, \bar{V}_1, \bar{V}_2, \bar{V}_3,$$

respectively. For instance, $|W_8| = |V_{3,5,\bar{6},7}| = |V_{6,2,\bar{3},7}| = |W_{16}|$. Finally, we observe that

$$|W_{25}| = |V_{1,\bar{2},4,\bar{6},\bar{7}} \cup V_{1,\bar{3},4,\bar{5},\bar{7}}|$$

$$= |V_{1,\bar{2},4,\bar{6},\bar{7}}| + |V_{1,\bar{3},4,\bar{5},\bar{7}}| - |V_{1,\bar{2},\bar{3},4,\bar{5},\bar{6},\bar{7}}|,$$

so that

$$|W_{25}| = |V_{1,\bar{2},4,\bar{6},\bar{7}}| + |V_{1,\bar{3},4,\bar{5},\bar{7}}| - |V_{1,\bar{2},4,\bar{5},\bar{7}}| \qquad \text{(by (2.3))}, \tag{2.4}$$

and

$$|W_{31}| = |V_{\bar{1},\bar{4}} \cup V_{\bar{2},\bar{6},\bar{7}} \cup V_{\bar{3},\bar{5},\bar{7}}|$$

$$= |V_{\bar{1},\bar{4}}| + |V_{\bar{2},\bar{6},\bar{7}}| + |V_{\bar{3},\bar{5},\bar{7}}|$$

$$- |V_{\bar{1},\bar{2},\bar{4},\bar{6},\bar{7}}| - |V_{\bar{1},\bar{3},\bar{4},\bar{5},\bar{7}}| - |V_{\bar{2},\bar{3},\bar{5},\bar{6},\bar{7}}| + |V_{\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7}}|$$

$$= |V_{\bar{1},\bar{4}}| + |V_{\bar{2},\bar{6},\bar{7}}| + |V_{\bar{3},\bar{5},\bar{7}}| - |V_{\bar{1},\bar{4},\bar{7}}| - |V_{\bar{1},\bar{4},\bar{7}}|$$

$$- |V_{\bar{2},\bar{5},\bar{7}}| + |V_{\bar{1},\bar{4},\bar{7}}| \qquad \text{(by (2.3))},$$

so that

$$\text{(2.5)} \qquad |W_{31}| = |V_{\bar{1},\bar{4}}| + |V_{\bar{2},\bar{6},\bar{7}}| + |V_{\bar{3},\bar{5},\bar{7}}| - |V_{\bar{1},\bar{4},\bar{7}}| - |V_{\bar{2},\bar{5},\bar{7}}|.$$

From these three observations one can easily deduce that, in order to compute asymptotic estimates for the number of pairs (a,b) in the sets $W_0, W_1, \ldots, W_{31}$, it is sufficient to determine these estimates only for the following twelve sets:

$$\text{(2.6)} \quad \begin{cases} V_{3,6},\ V_{3,\bar{4}},\ V_{3,\bar{5}},\ V_{\bar{1},\bar{4}},\ V_{\bar{2},\bar{6}},\ V_{\bar{2},\bar{5}}, \\ V_{3,5,\bar{6}},\ V_{3,4,\bar{5}},\ V_{1,\bar{3},\bar{4}}, \\ V_{2,\bar{3},5,\bar{6}},\ V_{1,\bar{2},4,\bar{6}},\ V_{1,\bar{2},4,\bar{5}}. \end{cases}$$

In order to save space we only give detailed computations for the three sets $V_{3,6}$, $V_{3,5,\bar{6}}$ and $V_{2,\bar{3},5,\bar{6}}$. The examples are fully illustrative for the other nine sets. The results are given in Table 3. This table also gives *all* sets which have, by the permutation property, the same number of elements as one of the twelve sets in (2.6).

TABLE 3

Asymptotic estimates of the number of elements in certain sets

set	estimate ($n\to\infty$)	set	estimate ($n\to\infty$)
$V_{3,6}$	$\sim \frac{1}{10} n^2$	$V_{3,5,\bar{6}}$, $V_{2,\bar{3},6}$	$\sim \frac{1}{40} n^2$
$V_{3,\bar{4}}$, $V_{\bar{1},6}$	$\sim \frac{1}{42} n^2$	$V_{3,4,\bar{5}}$, $V_{1,\bar{2},6}$	$\sim \frac{1}{56} n^2$
$V_{3,\bar{5}}$, $V_{\bar{2},6}$	$\sim \frac{1}{24} n^2$	$V_{1,\bar{3},\bar{4}}$, $V_{\bar{1},4,\bar{6}}$	$\sim \frac{5}{84} n^2$
$V_{\bar{1},\bar{4}}$	$\sim \frac{1}{12} n^2$	$V_{2,\bar{3},5,\bar{6}}$	$\sim \frac{1}{60} n^2$
$V_{\bar{2},\bar{6}}$, $V_{\bar{3},\bar{5}}$	$\sim \frac{5}{24} n^2$	$V_{1,\bar{2},4,\bar{6}}$, $V_{1,\bar{3},4,\bar{5}}$	$\sim \frac{9}{280} n^2$
$V_{\bar{2},\bar{5}}$	$\sim \frac{1}{6} n^2$	$V_{1,\bar{2},4,\bar{5}}$	$\sim \frac{1}{60} n^2$

$\underline{V_{3,6}}$. By (2.1), any element $(a,b) \in V_{3,6}$ satisfies $3b - 2a \le n$ and $3a - 2b \ge 1$, so that

$$a \ge \max\left(\frac{3b-n}{2}, \frac{2b+1}{3}\right) = \begin{cases} \dfrac{3b-n}{2}, & \text{if } b > \dfrac{3n+2}{5}, \\ \dfrac{2b+1}{3}, & \text{if } b \le \dfrac{3n+2}{5}. \end{cases}$$

It follows that if $b \le (3n+2)/5$, then $(2b+1)/3 \le a < b$, and if $b > (3n+2)/5$, then $(3b-n)/2 \le a < b$.

Hence,

$$|V_{3,6}| \sim \sum_{b=1}^{(3n+2)/5} (b-(2b+1)/3) + \sum_{b=(3n+2)/5}^{n} (b-(3b-n)/2)$$

$$\sim \sum_{b=1}^{3n/5} b/3 + \sum_{b=3n/5}^{n} (n-b)/2 \sim n^2/10.$$

$\underline{V_{3,5,\bar{6}}}$. By (2.1), any element $(a,b) \in V_{3,5,\bar{6}}$ satisfies $3b - 2a \le n$, $2a - b \ge 1$ and $3a - 2b < 1$, so that

$$\begin{cases} a \ge \max\left(\dfrac{3b-n}{2}, \dfrac{b+1}{2}\right) = \begin{cases} \dfrac{3b-n}{2}, & \text{if } b > \dfrac{n+1}{2}, \\ \dfrac{b+1}{2}, & \text{if } b \le \dfrac{n+1}{2}, \end{cases} & \text{and} \\ a < \dfrac{2b+1}{3}. \end{cases}$$

If $b > (n+1)/2$, then the conditions $a \ge (3b-n)/2$ and $a < (2b+1)/3$ make sense only if $(3b-n)/2 < (2b+1)/3$, so that $b < (3n+2)/5$. Furthermore, if $b \le (n+1)/2$, then $(b+1)/2 \le a < (2b+1)/3$.

Hence,

$$|V_{3,5,\bar{6}}| \sim \sum_{b=1}^{(n+1)/2} ((2b+1)/3-(b+1)/2) + \sum_{b=(n+1)/2}^{(3n+2)/5} ((2b+1)/3-(3b-n)/2)$$

$$\sim \sum_{b=1}^{n/2} b/6 + \sum_{b=n/2}^{3n/5} (n/2-5b/6) \sim n^2/40.$$

$V_{2,\bar{3},5,\bar{6}}$. By (2.1), any element $(a,b) \in V_{2,\bar{3},5,\bar{6}}$ satisfies $(2b-a) \leq n$, $3b - 2a > n$, $2a - b \geq 1$ and $3a - 2b < 1$, so that

$$\begin{cases} a \geq \max(2b-n,\ (b+1)/2) = \begin{cases} 2b - n, & \text{if } b > \dfrac{2n+1}{3}, \\ \dfrac{b+1}{2}, & \text{if } b \leq \dfrac{2n+1}{3}, \end{cases} \quad \text{and} \\ a < \min\left(\dfrac{3b-n}{2}, \dfrac{2b+1}{3}\right) = \begin{cases} \dfrac{3b-n}{2}, & \text{if } b \leq \dfrac{3n+2}{5}, \\ \dfrac{2b+1}{3}, & \text{if } b > \dfrac{3n+2}{5}. \end{cases} \end{cases}$$

The condition $b \leq (3n+2)/5$ implies that $b < (2n+1)/3$, so that if $b \leq (3n+2)/5$ we have $a < (3b-n)/2$ and $a \geq (b+1)/2$. This makes sense only if $(b+1)/2 < (3b-n)/2$, so that $b > (n+1)/2$. Furthermore, if $(3n+2)/5 < b \leq (2n+1)/3$, then $(b+1)/2 \leq a < (2b+1)/3$. Finally, if $b > (2n+1)/3$, then $2b - n \leq a < (2b+1)/3$. This makes sense only if $2b - n < (2b+1)/3$, so that $b < (3n+1)/4$.

Hence,

$$|V_{2,\bar{3},5,\bar{6}}| \sim \sum_{b=(n+1)/2}^{(3n+2)/5} ((3b-n)/2-(b+1)/2) + \sum_{b=(3n+2)/5}^{(2n+1)/3} ((2b+1)/3-(b+1)/2) + \sum_{b=(2n+1)/3}^{(3n+1)/4} ((2b+1)/3-(2b-n))$$

$$\sim \sum_{b=n/2}^{3n/5} (b-n/2) + \sum_{b=3n/5}^{2n/3} b/6 + \sum_{b=2n/3}^{3n/4} (n-4b/3) \sim n^2/60.$$

3. THE ASYMPTOTIC BEHAVIOUR OF $t_i(n)$

It is clear that $t_i(n)$, the number of pairs (a,b) with $1 \leq a,b \leq n (a \neq b)$, which belong to i integer arithmetic progressions of length four between 1 and n, inclusive, is twice the number of pairs for which $a < b$.

For $(a,b) \in \Omega_n$ we have $f_n(a,b) = 0$ if and only if (a,b) satisfies the condition $c_1 \wedge c_2 \wedge c_3 \wedge c_4 \wedge c_5 \wedge c_6$. Hence, $f_n(a,b) = 0$ if and only if $(a,b) \in W_{31} \cap \bar{V}_8$. From (2.5) and Table 3 it follows that

$$t_0(n) \sim 2\cdot\frac{2}{3}\cdot\left(\frac{1}{12}+\frac{5}{48}+\frac{5}{48}-\frac{1}{24}-\frac{1}{12}\right)\cdot n^2 = \frac{2}{9}n^2.$$

For $(a,b) \in \Omega_n$ we have $f_n(a,b) = 1$ if and only if (a,b) satisfies the condition

$$(\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge c_6) \vee (\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge c_5\wedge\bar{c}_6) \vee (\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge c_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee$$

$$(\bar{c}_1\wedge\bar{c}_2\wedge c_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (\bar{c}_1\wedge c_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (c_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6).$$

Hence, $f_n(a,b) = 1$ if and only if

$$(a,b) \in (W_{31}\cap V_8) \cup (W_{30}\cup W_{29}\cup W_{27}\cup W_{23}\cup W_{15})\cap\bar{V}_8),$$

so that

$$t_1(n) \sim 2\cdot\left(\frac{1}{6}\cdot\frac{1}{3}+\frac{2}{3}\cdot\left(\frac{1}{120}+\frac{5}{168}+\frac{5}{168}+\frac{1}{48}+\frac{1}{48}\right)\right)\cdot n^2 = \frac{9}{35}n^2.$$

For $(a,b) \in \Omega_n$, we have $f_n(a,b) = 2$ if and only if (a,b) satisfies the condition

$$(\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge c_5\wedge c_6) \vee (\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge c_4\wedge\bar{c}_5\wedge c_6) \vee (\bar{c}_1\wedge\bar{c}_2\wedge c_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge c_6) \vee$$

$$(\bar{c}_1\wedge c_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge c_6) \vee (c_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge c_6) \vee (\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge c_4\wedge c_5\wedge\bar{c}_6) \vee$$

$$(\bar{c}_1\wedge\bar{c}_2\wedge c_3\wedge\bar{c}_4\wedge c_5\wedge\bar{c}_6) \vee (\bar{c}_1\wedge c_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge c_5\wedge\bar{c}_6) \vee (c_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge c_5\wedge\bar{c}_6) \vee$$

$$(\bar{c}_1\wedge\bar{c}_2\wedge c_3\wedge c_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (\bar{c}_1\wedge c_2\wedge\bar{c}_3\wedge c_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (c_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge c_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee$$

$$(\bar{c}_1\wedge c_2\wedge c_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (c_1\wedge\bar{c}_2\wedge c_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (c_1\wedge c_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6).$$

Hence, $f_n(a,b) = 2$ if and only if

$$(a,b) \in ((W_{30} \cup W_{29} \cup W_{27} \cup W_{23} \cup W_{15}) \cap V_8) \cup$$
$$((W_{28} \cup W_{26} \cup W_{22} \cup W_{14} \cup W_{25} \cup W_{21} \cup W_{13} \cup W_{19} \cup W_{11} \cup W_7) \cap \bar{V}_8),$$

so that, using (2.4),

$$t_2(n) \sim 2\Big(\Big(\frac{1}{120} + \frac{5}{168} + \frac{5}{168} + \frac{1}{48} + \frac{1}{48}\Big) \cdot \frac{1}{3} +$$
$$\Big(\frac{1}{80} + \frac{1}{80} + \frac{9}{560} + \frac{9}{560} - \frac{1}{120} + \frac{1}{84} + \frac{1}{84}\Big) \cdot \frac{2}{3}\Big) \cdot n^2$$
$$= \frac{107}{630} n^2.$$

For $(a,b) \in \Omega_n$, we have $f_n(a,b) = 3$ if and only if (a,b) satisfies the condition

$$(\bar{c}_1 \wedge \bar{c}_2 \wedge \bar{c}_3 \wedge c_4 \wedge c_5 \wedge c_6) \vee (\bar{c}_1 \wedge \bar{c}_2 \wedge c_3 \wedge \bar{c}_4 \wedge c_5 \wedge c_6) \vee (\bar{c}_1 \wedge c_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \wedge c_5 \wedge c_6) \vee$$
$$(c_1 \wedge \bar{c}_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \wedge c_5 \wedge c_6) \vee (\bar{c}_1 \wedge \bar{c}_2 \wedge c_3 \wedge c_4 \wedge \bar{c}_5 \wedge c_6) \vee (\bar{c}_1 \wedge c_2 \wedge \bar{c}_3 \wedge c_4 \wedge \bar{c}_5 \wedge c_6) \vee$$
$$(c_1 \wedge \bar{c}_2 \wedge \bar{c}_3 \wedge c_4 \wedge \bar{c}_5 \wedge c_6) \vee (\bar{c}_1 \wedge c_2 \wedge c_3 \wedge \bar{c}_4 \wedge \bar{c}_5 \wedge c_6) \vee (c_1 \wedge \bar{c}_2 \wedge c_3 \wedge \bar{c}_4 \wedge \bar{c}_5 \wedge c_6) \vee$$
$$(c_1 \wedge c_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \wedge \bar{c}_5 \wedge c_6) \vee (\bar{c}_1 \wedge \bar{c}_2 \wedge c_3 \wedge c_4 \wedge c_5 \wedge \bar{c}_6) \vee (\bar{c}_1 \wedge c_2 \wedge \bar{c}_3 \wedge c_4 \wedge c_5 \wedge \bar{c}_6) \vee$$
$$(c_1 \wedge \bar{c}_2 \wedge \bar{c}_3 \wedge c_4 \wedge c_5 \wedge \bar{c}_6) \vee (\bar{c}_1 \wedge c_2 \wedge c_3 \wedge \bar{c}_4 \wedge c_5 \wedge \bar{c}_6) \vee (c_1 \wedge \bar{c}_2 \wedge c_3 \wedge \bar{c}_4 \wedge c_5 \wedge \bar{c}_6) \vee$$
$$(c_1 \wedge c_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \wedge c_5 \wedge \bar{c}_6) \vee (\bar{c}_1 \wedge c_2 \wedge c_3 \wedge c_4 \wedge \bar{c}_5 \wedge \bar{c}_6) \vee (c_1 \wedge \bar{c}_2 \wedge c_3 \wedge c_4 \wedge \bar{c}_5 \wedge \bar{c}_6) \vee$$
$$(c_1 \wedge c_2 \wedge \bar{c}_3 \wedge c_4 \wedge \bar{c}_5 \wedge \bar{c}_6) \vee (c_1 \wedge c_2 \wedge c_3 \wedge \bar{c}_4 \wedge \bar{c}_5 \wedge \bar{c}_6).$$

Hence, $f_n(a,b) = 3$ if and only if

$$(a,b) \in ((W_{28} \cup W_{26} \cup W_{22} \cup W_{14} \cup W_{25} \cup W_{21} \cup W_{13} \cup W_{19} \cup W_{11} \cup W_7) \cap V_8) \cup$$
$$((W_{24} \cup W_{20} \cup W_{12} \cup W_{18} \cup W_{10} \cup W_6 \cup W_{17} \cup W_9 \cup W_5 \cup W_3) \cap \bar{V}_8),$$

so that, using (2.4),

$$t_3(n) \sim 2\left[\left(\frac{1}{80}+\frac{1}{80}+\frac{9}{560}+\frac{9}{560}-\frac{1}{120}+\frac{1}{84}+\frac{1}{84}\right) \frac{1}{3} + \left(\frac{1}{120}+\frac{1}{20}+\frac{1}{112}+\frac{1}{112}\right) \frac{2}{3}\right]\cdot n^2$$

$$= \frac{3}{20}n^2.$$

Similarly, it follows that $f_n(a,b) = 4$ if and only if

$$(a,b) \in ((W_{24} \cup W_{20} \cup W_{12} \cup W_{18} \cup W_{10} \cup W_6 \cup W_{17} \cup W_9 \cup W_5 \cup W_3) \cap V_8) \cup (W_{16} \cup W_8 \cup W_4 \cup W_2 \cup W_1) \cap \bar{V}_8),$$

so that

$$t_4(n) \sim 2\left[\left(\frac{1}{120}+\frac{1}{20}+\frac{1}{112}+\frac{1}{112}\right) \frac{1}{3} + \left(\frac{1}{80}+\frac{1}{80}\right) \frac{2}{3}\right]\cdot n^2$$

$$= \frac{53}{630}n^2.$$

Furthermore, $f_n(a,b) = 5$ if and only if

$$(a,b) \in ((W_{16} \cup W_8 \cup W_4 \cup W_2 \cup W_1) \cap V_8) \cup (W_0 \cap \bar{V}_8),$$

so that

$$t_5(n) \sim 2\left[\left(\frac{1}{80}+\frac{1}{80}\right)\cdot \frac{1}{3} + \frac{1}{20}\cdot \frac{2}{3}\right]\cdot n^2 = \frac{1}{12}n^2.$$

Finally, $f_n(a,b) = 6$ if and only if $(a,b) \in W_0 \cap V_8$, so that

$$t_6(n) \sim 2\cdot \frac{1}{20}\cdot \frac{1}{3}\cdot n^2 = \frac{1}{30}n^2.$$

The limit of the average value a_n of f_n may be computed as follows. The number of integer arithmetic progressions of length four all of whose terms are between 1 and n, inclusive, is given by $\sum_{k=1}^{n} [(n-k)/3]$, which is asymptotic to $n^2/6$. Counting each such progression 12 times (once for each pair in it), we obtain

$$\sum_{\substack{1\le a,b\le n \\ a\ne b}} f_n(a,b) \sim 2n^2.$$

Hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2-n} \sum f_n(a,b) = 2.$$

This result provides a check of the values of $t_i(n)$, computed before, since $\sum f_n(a,b) = \sum_{i=0}^{6} it_i(n)$, so that we must have

$$2 = \lim_{n\to\infty} \frac{1}{n^2-n} \sum f_n(a,b) = \lim_{n\to\infty} \frac{1}{n^2-n} \sum_{i=0}^{6} it_i(n)$$

$$= \sum_{i=0}^{6} i \lim_{n\to\infty} (t_i(n)/(n^2-n)).$$

REFERENCES

[1] DRESSLER, R.E., *A note on arithmetic progressions of length four*, Math. Mag. 47 (1974) 31-34.